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Elliptic inflation: interpolating from natural inflation to R^2 -inflation

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ABSTRACT: We propose an extension of natural inflation, where the inflaton potential is a general periodic function. Specifically, we study elliptic inflation where the inflaton potential is given by Jacobi elliptic functions, Jacobi theta functions or the Dedekind eta function, which appear in gauge and Yukawa couplings in the string theories compactified on toroidal backgrounds. We show that in the first two cases the predicted values of the spectral index and the tensor-to-scalar ratio interpolate from natural inflation to exponential inflation such as R^2 - and Higgs inflation and brane inflation, where the spectral index asymptotes to $n_s = 1 - 2/N \simeq 0.967$ for the e-folding number $N = 60$. We also show that a model with the Dedekind eta function gives a sizable running of the spectral index due to modulations in the inflaton potential. Such elliptic inflation can be thought of as a specific realization of multi-natural inflation, where the inflaton potential consists of multiple sinusoidal functions. We also discuss examples in string theory where Jacobi theta functions and the Dedekind eta function appear in the inflaton potential.

KEYWORDS: Cosmology of Theories beyond the SM, Supergravity Models, Superstring Vacua

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Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | Elliptic inflation | 3 |
| 2.1 | Cnoidal inflation | 4 |
| 2.2 | Theta inflation | 6 |
| 2.3 | Inflation with Dedekind eta function | 9 |
| 3 | Possible UV completion | 11 |
| 3.1 | Field theoretic cases | 12 |
| 3.2 | Supergravity models with moduli stabilization | 13 |
| 3.3 | Potential with a gauge coupling depending on the Dedekind eta function | 17 |
| 3.4 | Super-Planckian decay constant | 18 |
| 4 | Discussion and conclusions | 18 |

1 Introduction

The cosmic microwave background (CMB) is isotropic to roughly one part in 10^5 , and its temperature anisotropies have the correlations beyond the horizon size at the last scattering. This strongly implies that our Universe experienced an accelerated cosmic expansion, namely, inflation, at an early stage of the evolution [1–5]. In the slow-roll inflation paradigm [6, 7], the quantum fluctuations of the inflaton become scalar-mode metric perturbations (i.e. density perturbations), while the quantum fluctuations of the graviton become tensor-mode metric perturbations [8], inducing primordial B -mode polarization of the CMB. If the primordial B -mode polarization is detected, it will provide a definitive proof of inflation, and determine the inflation scale to be around the GUT scale.

The inflaton field excursion during the last 50 or 60 e-foldings must be greater than or comparable to the Planck scale, $M_P \simeq 2.4 \times 10^{18}$ GeV, in order to generate large tensor mode perturbations within the reach of the current and future CMB experiments. There have been proposed many such large-field inflation models. Among them, there are two simple and therefore important inflation models: quadratic chaotic inflation [9] and natural inflation [10]. The natural inflation consists of a single cosine function and it asymptotes to the quadratic chaotic inflation in the limit of a large decay constant F . The Planck constraint on the tensor-to-scalar ratio r reads [11]

$$r < 0.11 \quad (95\% \text{ C.L.}), \quad (1.1)$$

and the quadratic chaotic inflation model is on the verge of being excluded, while some of the parameter space for the natural inflation ($F \lesssim 5M_P$) has been excluded.¹

There is one interesting class of inflation models where the predicted values of n_s and r sits near the best-fit values of the Planck observation: R^2 inflation [3] or Higgs inflation [14] (see also refs. [15–22] for related works). Of course this does not preclude any other models lying somewhere between the natural inflation and R^2 inflation. In fact, there are various extensions of the quadratic chaotic and natural inflation models, where a variety of values of n_s and r is realized. For instance, if one considers a polynomial chaotic inflation with the potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \kappa\phi^3 + \lambda\phi^4 \dots$, the value of r can be reduced to give a better fit to the Planck data [23–26]. Similarly, if there are multiple sinusoidal functions with different potential heights and decay constants, a variety of values of n_s and r can be realized [27–29], and such extension of the natural inflation is called *multi-natural inflation*. It was shown in ref. [27] that, even with two sinusoidal functions, the lower bound on the decay constant disappears because a small-field inflation (axion hilltop inflation) is realized for a certain choice of the parameters. Note that, although multiple sinusoidal functions are generated in the extra-natural inflation [30], it does not significantly improve the bound on the decay constant as the higher order terms are suppressed. It was recently pointed out that this problem can be circumvented by adding appropriate matter fields in the bulk [31], and this provides another UV completion of the multi-natural inflation.

In this paper we propose another extension of the natural inflation. The inflaton potential in natural inflation is given by a single sinusoidal function, which is a specific type of periodic function. A periodic potential is attractive from the model building point of view because it often appears when the flatness of the inflaton potential is protected by an approximate shift symmetry: the inflaton potential which arises from the breaking of a continuous shift symmetry respects the residual discrete one, leading to such periodic potential. We therefore focus on a class of inflation models with periodic potential. Specifically, we consider Jacobi elliptic functions, Jacobi theta functions, and the Dedekind eta function as the inflaton potential, and we call such inflation models based on elliptic functions and other related functions as *elliptic inflation*.

As an example of the elliptic inflation, we will first consider a Jacobi elliptic function as the inflaton potential:

$$V(\phi) = 2\Lambda^4 \left(1 - \text{cn}^2 \left(\frac{\phi}{2F}, k \right) \right), \quad (1.2)$$

where $\text{cn}(x, k)$ is the Jacobi elliptic cosine function with elliptic modulus $0 \leq k \leq 1$, and Λ and F are model parameters. We call the inflation model with the above potential *cnoidal inflation*. Interestingly, the cnoidal inflation is reduced to the natural inflation in the limit of $k \rightarrow 0$, while it is reduced to the exponential inflation including the brane inflation, R^2 inflation, and the Higgs inflation, in the limit of $k \rightarrow 1$.

¹The BICEP2 result [12] prefers a large r which is consistent with the quadratic chaotic inflation. However, the subsequent Planck polarization data [13] showed that the dust contamination in the BICEP2 region is larger than previously thought.

The Jacobi elliptic functions are closely related to the Jacobi theta functions, which often appear in theories with periodic boundary conditions. Periodicity or modular invariance play important roles in field theories in association with symmetries and dualities [32, 33]. In string theories, periodicity appears through compactifications of extra dimensions when a compact manifold has periodic properties. For instance, a holomorphic gauge coupling at one loop level is given by the Jacobi theta function $\vartheta_1(v, q)$ in toroidal compactification [34]. Here, v is a open string modulus (diagonal modes of the adjoint representation on D-branes), and $q = e^{i\pi\tau}$ is given by a complex structure of a torus τ in type IIB models (see also ref. [35] for their interpretation and further generalization). In our later example, the inflaton is identified with the real part of v . Another example is a Yukawa coupling among matter-like fields, which can be described by the Jacobi theta functions $\vartheta_3(v, q)$ in magnetized (or intersecting) D-brane models on tori [36, 37].² The theta functions appear in the gauge and Yukawa couplings because of the periodicity of the tori. Similar Yukawa couplings are obtained in Heterotic orbifolds through world-sheet instantons [39].³ Therefore, a scalar potential which depends on the theta functions appears naturally, and we shall show that it takes the following form,

$$V(\phi) \sim \left[\vartheta_1\left(\frac{\phi}{2\pi F}, q\right) \right]^a \left[\vartheta_3\left(\frac{\phi}{2\pi F}, q\right) \right]^b, \quad (1.3)$$

where a and b denote rational numbers. We call the inflation model with the above potential form *theta inflation*. We find that the theta inflation exhibits similar behavior to the cnoidal inflation.

It is possible to consider inflation models based on other special functions. In particular, inflation models with the Dedekind eta function was studied in ref. [41]. (See also ref. [42] for inflation models with modular functions.) Interestingly, the inflaton potential can be approximated by natural inflation with modulations, leading to a sizable running of the spectral index. In fact, it was shown in ref. [43] that a sizable running of the spectral index can be generated if there are small modulations on the inflaton potential. (See also refs. [44–50] for related work.) Here we will study the inflaton potential with the eta function in a greater detail.

The rest of this paper is organized as follows. In section 2 we study phenomenological aspects of the elliptic inflation, where we calculate the predicted values of n_s and r for the above three elliptic inflation models. In section 3 we discuss possible origins of the Jacobi theta functions and the Dedekind eta function in the string theories and implications for the large-field inflation. The last section is devoted to discussion and conclusions.

2 Elliptic inflation

In this section we first study two examples of the elliptic inflation. One is based on the Jacobi elliptic inflation (cnoidal inflation), and the other is based on the Jacobi theta

²In both IIB and IIA models on tori, it is possible to realize magnetized and intersecting D-brane system with any branes if one does not require the preservation of supersymmetry [38].

³In ref. [40], they considered a case in which Yukawa couplings are given by the Dedekind eta function of Kähler moduli with the target space modular invariance.

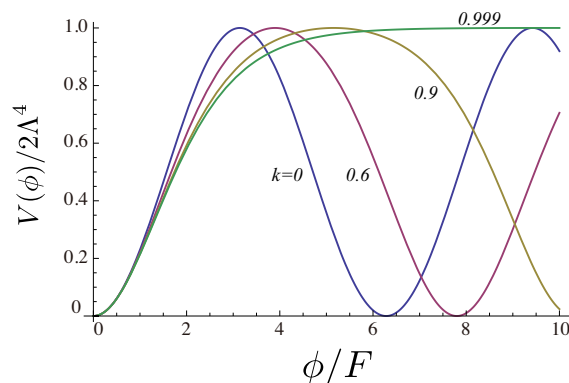


Figure 1. The inflaton potential $V(\phi)$ with the Jacobi elliptic function given by eq. (1.2) for $k = 0, 0.6, 0.9$ and 0.999 .

function (theta inflation). As we shall see below, they provide a one-parameter extension of the natural inflation, interpolating from natural inflation (approximately) to the R^2 -inflation. Finally we study the inflation based on the Dedekind eta function, and show that a sizable running of the spectral index can be generated because of the small modulations on the inflaton potential.

2.1 Cnoidal inflation

Let us consider the cnoidal inflation model with the potential eq. (1.2). The Jacobi elliptic functions $\text{sn}(x, k)$ and $\text{cn}(x, k)$ can be thought of as an extension of trigonometric functions, $\sin(x)$ and $\cos(x)$, respectively. In the limit of $k \rightarrow 0$, they become

$$\text{sn}(x, k) \rightarrow \sin x, \quad (2.1)$$

$$\text{cn}(x, k) \rightarrow \cos x, \quad (2.2)$$

while, in the limit of $k \rightarrow 1^-$,

$$\text{sn}(x, k) \rightarrow \tanh x, \quad (2.3)$$

$$\text{cn}(x, k) \rightarrow \text{sech } x. \quad (2.4)$$

In these limits, the inflaton potential $V(\phi)$ becomes

$$V(\phi) \rightarrow \begin{cases} \Lambda^4 \left(1 - \cos \frac{\phi}{F}\right) & \text{for } k \rightarrow 0, \\ 2\Lambda^4 \tanh^2 \frac{\phi}{2F} \simeq 2\Lambda^4 (1 - 4e^{-\phi/F}) & \text{for } k \rightarrow 1, \end{cases} \quad (2.5)$$

where we have assumed $\phi \gg F$ in the second equality. Therefore, the elliptic inflaton potential eq. (1.2) interpolates from natural inflation to the exponential inflation. See figure 1 for the inflaton potential $V(\phi)$ for several values of k . One can see from the figure that the potential becomes flatter around the potential maximum as k approaches unity.

To see how the predicted values of the spectral index n_s and the tensor-to-scalar ratio r change as a function of the model parameters, we numerically solve the inflaton equation

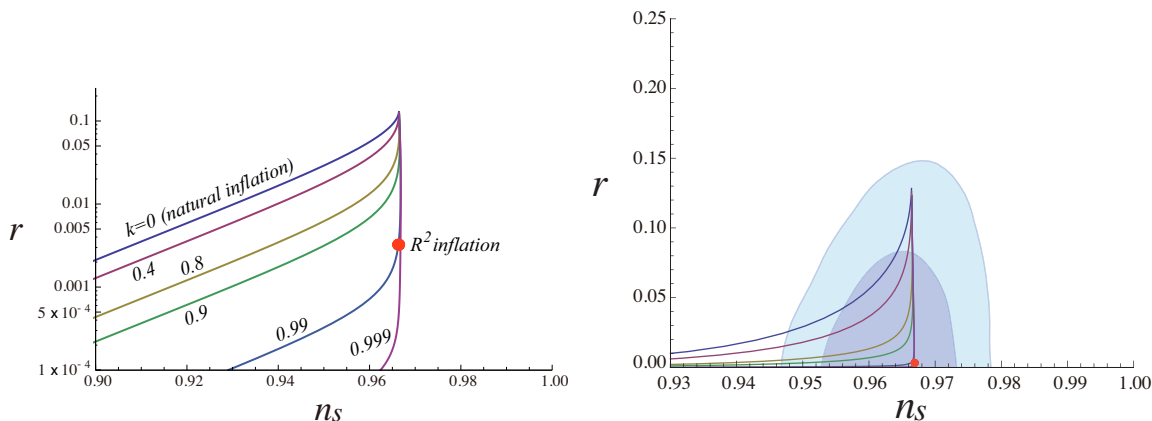


Figure 2. The spectral index and the tensor-to-scalar ratio in the elliptic inflation eq. (1.2) for $k = 0, 0.4, 0.8, 0.99$ and 0.999 from top to bottom with the e-folding number $N = 60$. The prediction of R^2 -inflation is shown by a red disc. We also overlay the Planck result (*Planck*+WP+BAO) [11] in the bottom panel for comparison. The elliptic inflation interpolates from natural inflation to R^2 -inflation as k varies from 0 to 1.

of motion,

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \quad (2.6)$$

with $V(\phi)$ given by eq. (1.2). The spectral index and the tensor-to-scalar ratio can be evaluated by the standard formulae,

$$n_s \simeq 1 + 2\eta - 6\varepsilon, \quad (2.7)$$

$$r \simeq 16\varepsilon, \quad (2.8)$$

where ε and η are the slow-roll parameters,

$$\varepsilon = \frac{1}{2} \left(\frac{V'(\phi)}{V(\phi)} \right)^2, \quad (2.9)$$

$$\eta = \frac{V''(\phi)}{V(\phi)}. \quad (2.10)$$

Here and in what follows we have adopted the Planck units, $M_P = 1$, and the prime denotes the derivative with respect to the inflaton field ϕ . In the actual numerical calculation, we have used a refined version of the above formulae including up to the second-order slow-roll parameters.

In figure 2 we show the calculated values of n_s and r for several values of k by varying F . One can see that the (n_s, r) predicted in the elliptic inflation indeed interpolates from natural inflation to exponential inflation. The latter includes the R^2 - or Higgs inflation, in which the potential is given by $V(\phi) \sim (1 - 2e^{-\sqrt{\frac{2}{3}}\phi})$. Here we have fixed the e-folding number to be $N = 60$. We also show in figure 3 how the n_s and r evolve as the decay constant F changes. For k sufficiently close to unity ($k \gtrsim 0.99$), the spectral index n_s stays around $1 - 2/N = 0.967$ for a wide range of the decay constant including even sub-Planckian values.

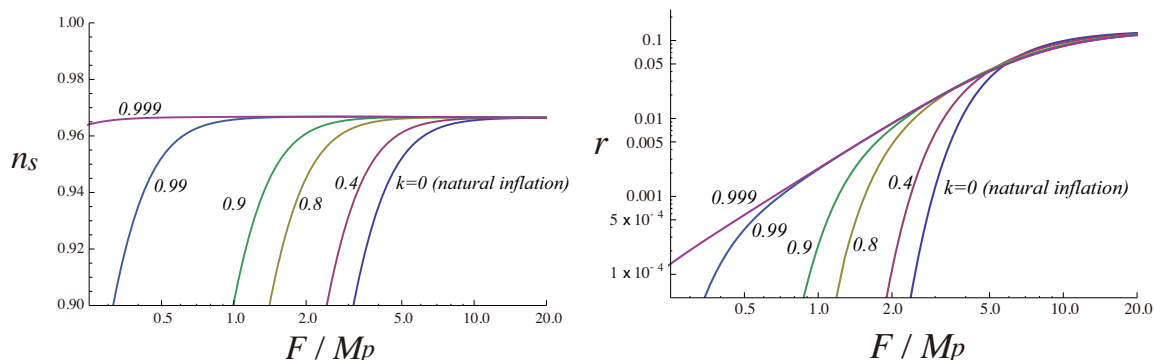


Figure 3. The dependence of n_s and r on the decay constant F in the cnoidal inflation eq. (1.2).

2.2 Theta inflation

Next we consider another example of the elliptic inflation, the theta inflation with the potential (1.3). It is known that the Jacobi elliptic functions are closely related to the Jacobi theta functions, $\vartheta_{0,1,2,3}(v, \tau)$. For instance, there are following relations:

$$\text{sn}(u, k) = \frac{\vartheta_3(0, q) \vartheta_1(v, q)}{\vartheta_2(0, q) \vartheta_0(v, q)}, \quad (2.11)$$

$$\text{cn}(u, k) = \frac{\vartheta_0(0, q) \vartheta_2(v, q)}{\vartheta_2(0, q) \vartheta_0(v, q)}, \quad (2.12)$$

where

$$k = \left(\frac{\vartheta_2(0, q)}{\vartheta_3(0, q)} \right)^2, \quad u = \pi v (\vartheta_3(0, q))^2, \quad (2.13)$$

or equivalently,

$$v = \frac{u}{2K(k)}, \quad q = e^{i\pi\tau}, \quad \tau = \frac{iK(\sqrt{1-k^2})}{K(k)}, \quad (2.14)$$

and $K(k)$ is the elliptic integral of the first kind. The Jacobi theta functions are given by

$$\begin{aligned} \vartheta_0(v, q) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2\pi i n(v + \frac{1}{2})} \\ &= 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2n\pi v, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \vartheta_1(v, q) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau (n + \frac{1}{2})^2 + 2\pi i (n + \frac{1}{2})(v + \frac{1}{2})} \\ &= 2 \sum_{n=0}^{\infty} (-1)^n q^{(n + \frac{1}{2})^2} \sin(2n + 1)\pi v, \end{aligned} \quad (2.16)$$

$$\begin{aligned}\vartheta_2(v, q) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau \left(n + \frac{1}{2}\right)^2 + 2\pi i \left(n + \frac{1}{2}\right)v} \\ &= 2 \sum_{n=0}^{\infty} q^{\left(n + \frac{1}{2}\right)^2} \cos(2n + 1)\pi v, \end{aligned} \quad (2.17)$$

$$\begin{aligned}\vartheta_3(v, q) &= \sum_{n=-\infty}^{\infty} e^{\pi i \tau n^2 + 2\pi i n v} \\ &= 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos 2n\pi v. \end{aligned} \quad (2.18)$$

Here, $|q| < 1$, i.e., $\text{Im}(\tau) > 0$. The Fourier series converge very fast for $|q| \ll 1$ as the higher order terms are exponentially suppressed.

The Jacobi theta functions appear in gauge and Yukawa couplings in string theories with toroidal compactifications. As we shall see in the next section, the Jacobi theta functions appear naturally in the scalar potential.

To be specific, let us consider the following inflaton potential,

$$V(\phi) = \frac{\Lambda^4}{2q} \left(\vartheta_3(0, q) - \vartheta_3\left(\frac{\phi}{2\pi F}, q\right) \right) \quad (2.19)$$

which corresponds to eq. (1.3) with $a = 0$ and $b = 1$. Note that a similar inflaton dynamics is obtained for the potential with the other theta functions because of the identities between the theta functions: $\vartheta_3(v + 1/2, q) = \vartheta_0(v, q)$, $\vartheta_3(v + \tau/2, q) = e^{-i\pi(\tau/4 + v)}\vartheta_2(v, q)$, $\vartheta_2(v + 1/2, q) = -\vartheta_1(v, q)$ and so on. In the limit of $q \rightarrow 0$, the above potential is reduced to the natural inflation since the higher order terms in the Fourier series are suppressed.

$$V(\phi) \rightarrow \Lambda^4 \left(1 - \cos \frac{\phi}{F} \right) \quad \text{for } q \rightarrow 0. \quad (2.20)$$

On the other hand, for $|q| \rightarrow 1$, many sinusoidal functions add up so as to make the inflaton potential extremely flat around the potential maximum. This is essentially same as the axion hilltop inflation where the inflaton potential becomes flat due to cancellation among multiple sinusoidal functions [27–29]. In this sense, the theta inflation can be thought of as a specific realization of the multi-natural inflation. (Also note that the Fourier series converge fast unless $|q|$ is extremely close to unity.) We show the inflaton potential for $q = 0, 0.7, 0.8$ and 0.9 in figure 4. As one can see from the figure, the inflaton potential around the local maximum becomes flatter as q approaches unity.

We similarly solve the inflaton dynamics with inflaton potential (2.19), and estimated n_s and r . The results are shown in figures 5 and 6. The spectral index n_s stays around 0.967 for sub-Planckian values of F , if $q \gtrsim 0.7$. It is interesting to note that the predicted values of n_s and r become closer to the R^2 -inflation as q becomes larger, but they do not exactly matches with the R^2 -inflation: the spectral index tends to be slightly smaller for the same value of r .

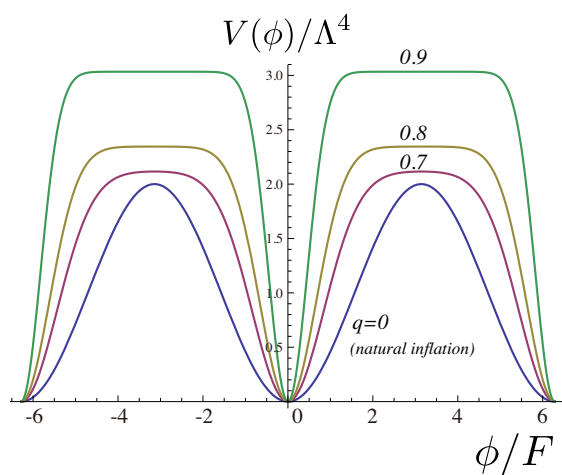


Figure 4. The inflaton potential $V(\phi)$ with the Jacobi theta function given by eq. (2.19) for $q = 0, 0.7, 0.8$ and 0.9 from bottom to top.

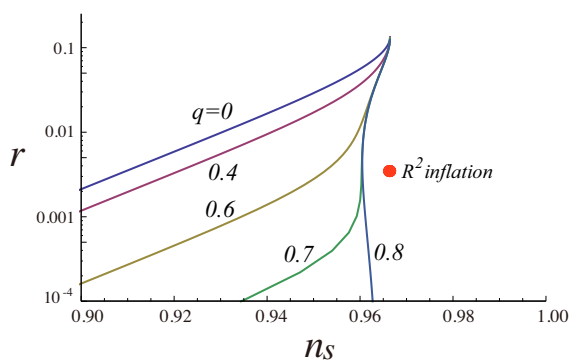


Figure 5. (n_s, r) for the theta inflation with the potential (2.19).

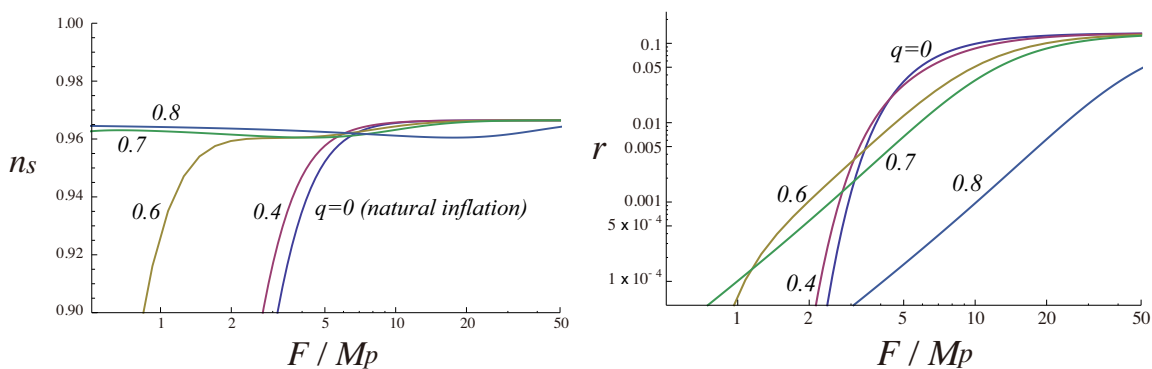


Figure 6. Same as figure 3 but for the theta inflation (2.19).

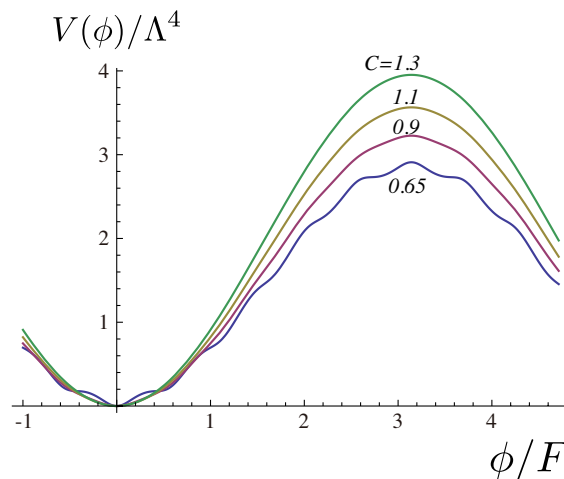


Figure 7. The inflaton potential with the Dedekind eta function (2.22) for $C = 1.3, 1.1, 0.9$ and 0.65 from top to bottom.

2.3 Inflation with Dedekind eta function

Now we study an inflation model with the Dedekind eta function,

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{n=1}^{\infty} (1 - q^{2n}), \quad (2.21)$$

where $q = e^{i\pi\tau}$ and $\text{Im}(\tau) > 0$. It is known that the eta function is given by the special values of the theta functions, for instance, $\eta^3(\tau) = \frac{1}{2}\vartheta_2(0, \tau)\vartheta_3(0, \tau)\vartheta_0(0, \tau)$.

To be specific, we will study the inflaton potential:

$$\begin{aligned} V(\phi) &= -\Lambda^4 \left[\eta^{-2} \left(\frac{6}{\pi F} \phi + iC \right) + \text{c.c.} \right] + \text{const.}, \\ &\propto \cos \frac{\phi}{F} + e^{-2\pi C} \cos \left(11 \frac{\phi}{F} \right) + \dots, \end{aligned} \quad (2.22)$$

where C is a real parameter, and the constant term is added so as to make the inflaton potential vanish at the origin, $V(0) = 0$. In the second line, we have expanded the eta function in terms of $|q| = e^{-\pi C} \ll 1$. The inflaton potential (2.22) is shown in figure 7 for several values of C . One can see from the figure that the inflaton potential has a maximum at $\phi = \pi F$ and it receives modulations for $C \lesssim 1$. Here, $(\phi, C) = (0, 1)$ implies the self-dual point under the modular transformation of $\tau \rightarrow -1/\tau$, i.e., $C \rightarrow 1/C$. For larger values of C , the above potential asymptotes to the natural inflation with the decay constant F , as one can see from the expansion of the inflaton potential. Note that the normalization of the potential height is chosen for visualization purpose. This does not affect the predicted values of r , n_s and its running in the following analysis.

The predicted values of n_s and r are shown in figure 8 for $C = 1.5, 1$ and 0.85 , where we have varied the value of the decay constant F , and set the e-folding number N to be 60. For $C = 1$ and $C = 0.85$, the predicted values of (n_s, r) rotate counterclockwise as

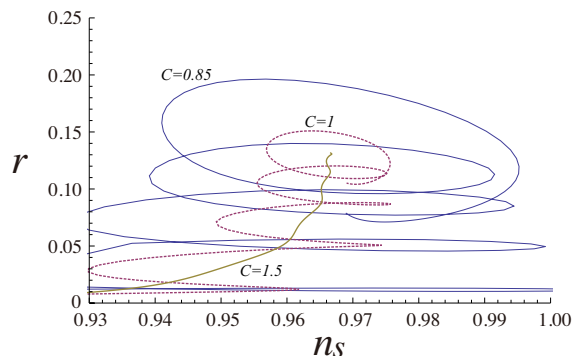


Figure 8. (n_s, r) for the inflation potential with the Dedekind eta function (2.22).

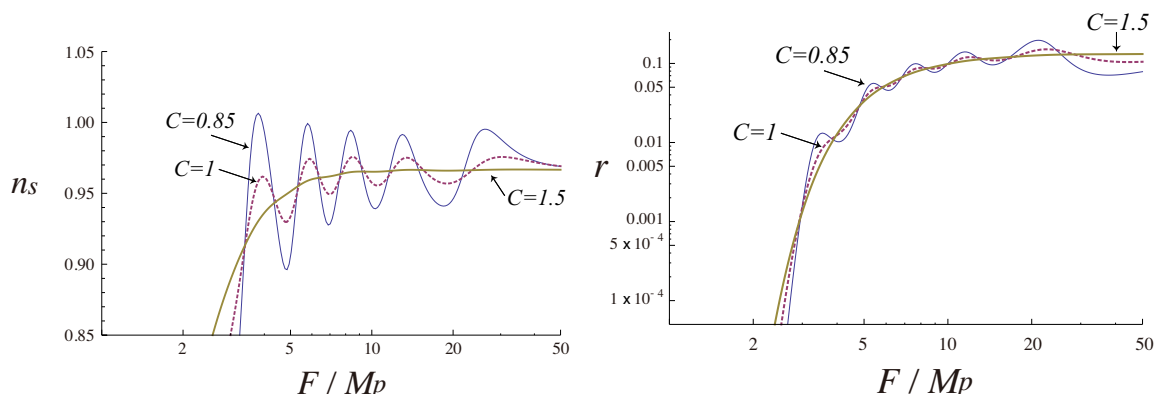


Figure 9. Same as figure 3 but for the inflaton potential (2.22).

F decreases. In contrast to the previous two cases, this model does not asymptote to the exponential inflation. The dependence of n_s and r on the decay constant F is shown in figure 9, and we find that the super-Planckian decay constant is required to be consistent with the Planck data. We will return to the issue of super-Planckian decay constant at the end of this section.

Finally let us evaluate the running of the spectral index, $dn_s/d\ln k$. Recently the inflation model with the Dedekind eta function was shown to generate a sizable running of the spectral index [41]. The running can be expressed in terms of the slow-roll parameters as

$$\frac{dn_s}{d\ln k} \simeq -24\varepsilon^2 + 16\varepsilon\eta - 2\xi, \quad (2.23)$$

where

$$\xi \equiv \frac{V'(\phi)V'''(\phi)}{V(\phi)^2}. \quad (2.24)$$

The current constraint on the running by *Planck* and WMAP polarization reads [51]

$$\frac{dn_s}{d\ln k} = -0.013 \pm 0.009 \quad (68\%). \quad (2.25)$$

For ε and η of $O(0.01)$, the first two terms in eq. (2.23) are of $O(10^{-3})$, one order of magnitude smaller than the center value of the current bound. In order to generate a

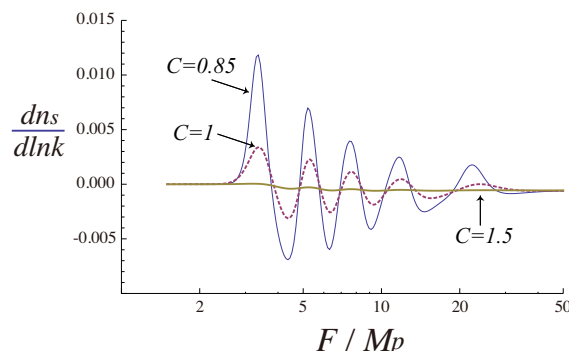


Figure 10. The running of the spectral index $dn_s/d \ln k$ as a function of the decay constant F in the inflation model (2.22).

sizable running of order $O(0.01)$, therefore, ξ must be large. For a simple class of inflaton potentials expressed by Taylor series with the finite truncation, however, a large negative running that is more or less constant over the observed cosmological scales would quickly terminate inflation within the e-folding number $N \lesssim 30$ [52]. There is a loophole in this analysis, and it was pointed out in ref. [43] that a sizable and almost constant running over the CMB scales $\Delta N_{\text{CMB}} \sim 8$ can be generated in an inflaton potential with small modulations, because the modulations can give a dominant contribution to V''' , while V , V' and V'' are not significantly modified. The running of the spectral index as a function of the decay constant is shown in figure 10. For slightly smaller values of C , $|dn_s/d \ln k|$ can be larger than 0.01, and it is more or less constant over the CMB scales.

3 Possible UV completion

We have introduced the Jacobi elliptic functions and theta functions in the scalar potential to extend the natural inflation, motivated by the recent Planck data. We have shown that they predict the spectral index and the tensor-to-scalar ratio interpolating from natural inflation to exponential inflation including the R^2 - or Higgs inflation. Such functions indeed appear in the string theories with extra dimension containing tori. As noted already, gauge and Yukawa interactions can be written in terms of the Jacobi theta functions in toroidal compactifications in intersecting or magnetized D-brane system [34, 36, 37]. Also on the magnetized orbifolds, e.g., $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and $T^2 \times T^4/\mathbb{Z}_2$, Yukawa couplings are similarly expressed with the theta functions [53]. This type of function comes from the wave functions of the fields on the torus, which respects the periodicity, in magnetized brane models (or summation of world-sheet instantons in intersecting brane models).⁴ So, the scalar potential with the Jacobi theta function will appear naturally in the low energy scales.

In passing, we note that the theta functions can also appear in scattering amplitudes at one-loop level of string world-sheet together with the Dedekind eta function (involving

⁴The wave functions are invariant under the periodic shift on the torus up to gauge transformation, which is represented by double pseudo periodicity of the Jacobi theta functions.

moduli integrals) [54]. This is similar in the field theories [55]. Also the eta function can appear in one-loop threshold corrections to gauge couplings after integrating out heavy Kaluza-Klein modes in the extra dimension [56]. Note that the eta function is given by the special value of the theta functions, as we have seen before.

In the following, we discuss several possibilities to generate the scalar potential with the theta functions. The inflaton candidate is an open string modulus on the D-branes, and hence it originates from the Wilson lines, positions of D-branes or their linear combination. For instance, if a four cycle \mathcal{S} in a Calabi-Yau, wrapped by D7-branes, has the Hodge numbers with $h_-^{1,0}(\mathcal{S}) \neq 0$ or $h_-^{2,0}(\mathcal{S}) \neq 0$, there appear $h_-^{1,0}(\mathcal{S})$ Wilson line moduli or $h_-^{2,0}(\mathcal{S})$ position moduli, respectively [57]. To discuss whether such models are really viable for inflation, however, it is necessary to construct an explicit model of the moduli stabilization, supersymmetry (SUSY) breaking, as well as the embedding of the standard model sector etc. The open string inflation and the related issues have been studied extensively in the literature. The purpose of this section is to present several possible scenarios where the Jacobi theta functions appear in the scalar potential, motivated by the recent Planck results on n_s and r .

3.1 Field theoretic cases

As a toy model, we first consider a case where the Yukawa coupling is given by the theta function $\vartheta_3(v, q)$, where $q = e^{i\pi\tau}$, and τ is a torus complex structure modulus τ in magnetized brane model⁵ (or the volume modulus and B-field in the intersecting brane system), and v is identified with the inflaton $\phi/2\pi F$. In a more realistic scenario, one has to worry about the stabilization of the imaginary component of v , which we shall discuss in the next subsection.⁶

Now the question is how to generate a scalar potential which depends on the Yukawa coupling. One possibility is to use a strong dynamics:

$$\begin{aligned} \mathcal{L} &= \vartheta_3(v, q) \varphi \psi^c \psi \\ \rightarrow V &= \vartheta_3(v, q) \langle \varphi \rangle \langle \psi^c \psi \rangle \equiv \Lambda^4 \vartheta_3\left(\frac{\phi}{2\pi F}, q\right), \end{aligned} \quad (3.1)$$

where φ and $\psi + \psi^c$ are scalar and vector-like fermion fields in the strongly coupled hidden sector respectively. Here, we have assumed that the scalar develops a non-zero vacuum expectation value (VEV) and the fermion pair condensates through the strong dynamics.

⁵A Yukawa coupling consists of a product of theta functions if a compact extra dimension is given by a product of tori. We will focus on the contribution from a single torus and neglect contributions from the other tori, assuming that other moduli become heavy owing to flux compactification. One may consider $T^4/\mathbb{Z}_2 \times T^2$, in which Wilson line moduli could be required to explain the smallness of the observable Higgs mass, although the moduli are not necessary to explain Yukawa textures [53]. On the other hand, ϑ_3 is often used for realizing a large (top) Yukawa coupling because it contains unity and ϑ_2 is used for a smaller top Yukawas; see eq. (2.18).

⁶Note also that theta inflation with $|q| \sim 1$ would require a small complex structure, i.e., $\text{Im}(\tau) \ll 1$. Then we may have a difficulty to control the coupling expansion, because then a complex structure on a torus can be translated into the volume under T-duality. Such a problem may be avoided, if q in the theta function depends on a linear combination of moduli or a large volume extra dimension is realized with stabilized moduli, compensating small values of complex structure moduli.

In a supersymmetric case, we may consider the IYIT model [58, 59], where there are four superfields of Ψ_j ($i, j = 1, 2, 3, 4$) with fundamental representation under $SU(2)$ gauge theory and six singlet superfields $\Phi^{ij} = -\Phi^{ji}$. The low energy superpotential becomes

$$W = \vartheta_3(v, q) \Phi^{ij} [\Psi_i \Psi_j] \quad (3.2)$$

with $\text{Pf}([\Psi_i \Psi_j]) = \Lambda_{SU(2)}^4$. Here, $[\Psi_i^c \Psi_j]$ is a composite operator and $\Lambda_{SU(2)}$ is dynamically generated scale under the $SU(2)$ theory. Thus, scalar potential is given by the F-component of Φ :

$$V = \Lambda_{SU(2)}^4 |\vartheta_3(v, q)|^2 + \mathcal{O}(\Phi). \quad (3.3)$$

Here, we have assumed that $\Phi = 0$ is the minimum owing to the Kähler potential $K = |\Phi|^2 - c|\Phi|^4/\Lambda_{SU(2)}^2$, in which $c = \mathcal{O}(10^{-2})$ is obtained after integrating out heavy modes with the masses of order $\Lambda_{SU(2)}$. Alternatively, one may consider the trilinear scalar term of the Yukawa interaction, assuming the scalars develop non-zero VEVs by some dynamics. In this case the scalar potential is approximately given by $V \sim m_{3/2} \vartheta_3(v, q) \langle \Phi \rangle^3$.

3.2 Supergravity models with moduli stabilization

In a more realistic stringy set-up one has to consider both real and imaginary components of v . In order to realize an effective single-field inflation, one needs to ensure that the other degrees of freedom including the imaginary component of v are stabilized with a mass heavier than (or comparable to) the Hubble parameter during inflation. To see this, let us consider a supergravity model in type IIB orientifold with flux compactification [60–67].

The following is the simplified effective action

$$K = K(S + \bar{S}; T + \bar{T}) - 3 \log[-i(\tau - \bar{\tau})] + \sum_{\alpha} F_{\alpha}^2 |v_{\alpha} - \bar{v}_{\alpha}|^2 + \cdots, \quad (3.4)$$

$$W = \int G \wedge \Omega(S, \tau, v_{\alpha}) + A(S, \tau, v_{\alpha}, \Psi) e^{-aT} + W_{\text{hidden Yukawa}}(v_{\alpha}, \tau, \Psi) + \cdots, \quad (3.5)$$

where S is a dilaton-axion, T is a Kähler modulus, τ is a complex structure modulus, and v_{α} is a open string modulus. F_{α} is a decay constant for open string modulus, e.g., $F_W^2 = 3/(T + \bar{T})$ and $F_P^2 = 1/(S + \bar{S})$ for D7 Wilson line moduli and D7 position moduli, respectively [57], and they are expected to be comparable to the string/Planck scale. We shall return to the issue of the size of the decay constant later. G is the three form flux, Ω is holomorphic three form and the second term in the superpotential arises from instantons or gaugino condensations. We have included trilinear superpotential $W_{\text{hidden Yukawa}}$ between matter-like fields Ψ_i in the hidden sector. In the Kähler potential, the ellipsis include the terms for Ψ_i and SUSY breaking sector and the quantum corrections [68] in addition to the Standard Model (SM) fields. The ellipsis in the superpotential contains contributions from SUSY-breaking and the SM sector.

With the above effective action, τ and S can be stabilized by a choice of three form fluxes G : $\int G \wedge \Omega \supset \tau^2 + S\tau$. We may obtain a light open string moduli v , a certain combination of v_{α} , if

$$\frac{\partial}{\partial v} \int G \wedge \Omega(S, \tau, v_{\alpha}) = 0. \quad (3.6)$$

The other degrees of freedom in $\{v_\alpha\}$ are assumed to be stabilized with a heavy mass by $\int G \wedge \Omega \supset v_\alpha^2$, and therefore can be integrated out in the low energy. For instance, the Wilson line moduli are not included in flux-induced superpotential because of the internal gauge symmetry [67]. Alternatively, only a part of the position moduli on D-branes may be fixed by a selection of the closed string fluxes, magnetic (worldvolume) fluxes and geometry through the tadpole condition [69]. (Similar argument can be applied also to the complex structure moduli [41, 70] and to type IIA moduli stabilization without metric fluxes [71].) Finally, through instantons or gaugino condensations on the stack of branes, the Kähler moduli T can be stabilized taking account of the SUSY breaking. The SUSY breaking also stabilizes the remaining moduli and produces the scalar potential of the Jacobi theta function through a gaugino mass or a trilinear scalar term. Here, we expect that $\text{Im}(v)$ obtains the gravitino mass $m_{3/2} \propto \int G \wedge \Omega$ via the SUSY-breaking effect because the Kähler potential depends only on the imaginary part of v as in eq. (3.5) at the tree level. Thus, $\text{Re}(v) \equiv \phi/2\pi F$ will be the lightest degrees of freedom, a good candidate for the inflaton.

In the following we consider various ways to generate the inflaton potential, which depends on the Jacobi theta functions, $\vartheta_i(v, q)$. We will discuss also an issue of one-loop breakdown of the shift symmetry along $\text{Re}(v)$.

Case 1: gauge coupling depending on the theta function. Let us first focus on the dependence on v through a gauge coupling in the strongly coupled hidden sector. This v -dependence appears in the non-perturbative term as we shall see below.

On the stack of N D7-branes, the holomorphic gauge coupling at one-loop level contains the Jacobi theta function in the presence of e.g. a stack of n D3-branes [34]:

$$f_{\text{one-loop}} \supset -\frac{n}{2\pi} \log(\vartheta_1(v, q)), \quad (3.7)$$

where $q = e^{i\pi\tau}$ is given by a torus complex structure modulus τ and we have assumed that the gauge coupling depends on v , and the tree level coupling is given by $f_{\text{tree}} = T$. This is regarded as a backreaction from n D3-branes to the worldvolume of N D7-branes [35]. In this case v denotes the position of D3-branes on the torus. When the N D7-branes undergo gaugino condensation,⁷ the second non-perturbative term in eq. (3.5) becomes

$$W_{\text{non-pert}} \propto \exp \left[-\frac{2\pi}{N} \left(T - \frac{n}{2\pi} \log(\vartheta_1(v, q)) \right) \right] \quad (3.8)$$

$$\equiv \Lambda_{\text{SU}(N)}^3 (\vartheta_1(v, q))^{\frac{n}{N}}. \quad (3.9)$$

Here, $\Lambda_{\text{SU}(N)}$ is the dynamically generated scale under the pure $\text{SU}(N)$ super Yang-Mills theory. Now we have the gaugino mass term $m_{3/2}\lambda\lambda$ in the $\text{SU}(N)$ sector, the scalar

⁷Similar type of the superpotential has been observed through D-instanton contributions in refs. [72, 73] with $\text{SL}(2, \mathbb{Z})$ duality in type IIB supergravity compactified on Calabi-Yau orientifold with odd parity Kähler moduli \mathcal{G} : for instance, $W \sim \theta_m(\mathcal{G}, iS) \eta^{-10}(iS)$ may be generated by D-instantons. Here, $\theta_m(\mathcal{G}, iS) = \sum_{n \in L+r} e^{-Sn^2/2} e^{im\mathcal{G}n}$ is the theta function of weight $s/2$ with index m . L is some positive definite rational lattice of dimension s , and r is some vector which admits an expansion in a basis of L with rational coefficients. Then, we have $\partial_{\mathcal{G}} \int G \wedge \Omega = 0$, hence $\text{Re}(\mathcal{G})$ can be an inflaton candidate if exists, because there will be just $\text{Im}(\mathcal{G})$ dependence in the Kähler potential at a perturbative level. This looks similar to the superpotential discussed in inflation model with Dedekind eta function.

potential is given by

$$V \sim m_{3/2} \Lambda_{\text{SU}(N)}^3 (\vartheta_1(v, q))^{\frac{n}{N}} + \mathcal{O}(\Lambda_{\text{SU}(N)}^6), \quad (3.10)$$

where we neglected the presence of matter Ψ_i and higher order terms in $\Lambda_{\text{SU}(N)}$. The inflaton is identified with the real component of v , the position of D3-branes: $\text{Re}(v) \equiv \phi/2\pi F$. Thus, the inflaton potential depending on ϑ_1 arises from the interactions between n D3 branes and N D7 branes through gaugino condensation in this case.

If the inflation is driven by the above inflaton potential, the Hubble scale during the inflation will be given by $H_{\text{inf}}^2 \sim m_{3/2} \Lambda_{\text{SU}(N)}^3$. In the KKLT case [60], we expect $\Lambda_{\text{SU}(N)}^3 \lesssim m_{3/2} \lesssim \Lambda_{\text{SU}(N)}$, and so, $H_{\text{inf}} \lesssim m_{3/2}$. Therefore we may be able to avoid the decompactification problem [74]. The potential with the double gaugino condensations can ameliorate this problem. (See also discussions in ref. [75].) The scalar potential induced by a similar mechanism was used to drive slow-roll inflation in the context of the warped brane inflation [35, 76–79], where inflation is driven by moving D3-branes in the warped geometry with anti D3-branes at the tip. On the other hand, here we consider the inflation with a D-brane modulus in the bulk extra dimension, and the inflaton potential arises from the interactions between D3-branes and D7-branes. We may use anti D3-branes at the tip of a warped throat to uplift the AdS vacuum to the dS one with a small and positive cosmological constant. The interaction between branes and anti branes can be neglected if it is a sufficiently strongly warped throat. Alternatively, we may consider a field-theoretic spontaneous SUSY breaking for uplifting [80–83]. In either case the uplifting sector is required to preserve an approximate shift symmetry along inflaton direction [76] for successful slow roll inflation.

Lastly, we note that there are v -dependent one-loop corrections in the Kähler potential [68], which induce the mass term for the inflaton $V \sim c m_{3/2}^2 \phi^2$ with an expectation of $c = \mathcal{O}(10^{-2})$ or so. For successful slow-roll inflation, such corrections must be sufficiently small, $c \lesssim 10^{-2} (\Lambda_{\text{SU}(N)}^3 / m_{3/2})$.

Case 2: Yukawa coupling depending on the theta function. As noted already, Yukawa couplings are given by the Jacobi theta function in the toroidal compactification of extra dimension with magnetic fluxes in the string theory. The superpotential is given by

$$W = \vartheta_3(v, q) \Psi_i \Psi_j \Psi_k, \quad (3.11)$$

where $\Psi_{i,j,k}$ are hidden matter-like superfields localized by the magnetic fluxes on the extra dimension, v is a open string modulus and q is a torus complex structure moduli. All the moduli other than v are assumed to be stabilized with a heavy mass. Taking account of the SUSY breaking effects, there will be a SUSY-breaking trilinear scalar term: $V_{\text{SUSY-breaking}} = m_{3/2} \vartheta_3(v, q) \Psi_i \Psi_j \Psi_k + \text{c.c.}$. (Here, note that $\Psi_{i,j,k}$ are scalar component of the respective superfields.) The scalars can develop non-zero VEVs, for instance, via D-term potentials involving with Fayet-Iliopoulos term ξ generated by the magnetic fluxes and stabilized T : $D \propto \xi + \sum_i q_i |\Psi_i|^2$, where q_i is an U(1) charge of Ψ_i . See, e.g., refs. [64, 84–88] for moduli stabilization with D-term potential. Then the potential becomes:

$$V \sim a \vartheta_3(v, q) + b (\vartheta_3(v, q))^2 + \mathcal{O}(\Psi^5). \quad (3.12)$$

Here, $\text{Re}(v) \equiv \phi/2\pi F$ is a candidate for the inflaton, $a = \mathcal{O}(m_{3/2}\Psi^3)$, $b = \mathcal{O}(\Psi^4)$, and we have added SUSY F-term contribution $V_{\text{SUSY}} \propto |\partial_\Psi W|^2$, and neglected supergravity contributions of $\mathcal{O}(\Psi^5)$, which contain derivatives of the theta function. Whether decompactification occurs depends the VEVs of $\{\Psi_i\}$, and $a, b \lesssim m_{3/2}^2$ is required. As discussed in the previous example, we need to suppress one-loop corrections to the inflaton mass for successful slow-roll inflation: $c \lesssim 10^{-2}a/m_{3/2}^2$ or $c \lesssim 10^{-2}b/m_{3/2}^2$. Here c is a coefficient of the one-loop induced inflaton mass term normalized by the gravitino mass squared.

Case 3: Yukawa coupling depending on the theta function and strong dynamics.

Now we consider a scenario similar to the case 2, but Ψ_i are stabilized by a strong dynamics as in refs. [64, 89]:

$$W_{\text{non-pert}} = (N_c - N_f) \left(\frac{Ae^{-2\pi T}}{\det(\Psi^c \Psi)} \right)^{\frac{1}{N_c - N_f}}, \quad (3.13)$$

$$W_{\text{hidden Yukawa}} = \vartheta_3(v, q) \Phi \Psi^c \Psi, \quad (3.14)$$

where A includes ϑ_1 -dependent term in the non-perturbative effect as discussed in the case 1, $\Psi^c + \Psi$ are vector-like superfields charged under $\text{SU}(N_c)$ gauge group with the number of $N_f (< N_c)$ flavors, and Φ is a singlet superfields. Note that there are no open string moduli (adjoint modes) in the strong coupling brane to realize an asymptotically freedom. Here, let us consider three stacks of D-branes labeled by (α, β, γ) for simplicity;

$$v = I_\alpha v_\alpha + I_\beta v_\beta + I_\gamma v_\gamma \quad (3.15)$$

is a linear combination of open string moduli $(v_\alpha, v_\beta, v_\gamma)$ on the respective branes [36, 37], and $\text{Re}(v) \equiv \phi/2\pi F$ is the inflaton. Here, $I_\alpha = \text{Ind}(D)_{\beta\gamma}$, $I_\beta = \text{Ind}(D)_{\gamma\alpha}$, and $I_\gamma = \text{Ind}(D)_{\alpha\beta}$, where $\text{Ind}(D)_{\alpha\beta} \in \mathbb{Z}$ is the index of Dirac operator, which depends on magnetic flux between the α - and β -th D-brane and so on.⁸ Hence $I_\alpha = -I_\beta = N_f$ and I_γ counts the number of singlet Φ . Note that $v_\gamma \equiv 0$ when there exists the strongly coupled gauge theory on the γ -th D -brane, because we have assumed there is no open string moduli on γ -th D -brane. Note that q is given by $q = \exp(i\pi\tau|I_\alpha I_\beta I_\gamma|)$, where τ is a complex structure moduli on the torus.

Here, we shall consider stabilization of matter-like fields. Φ can be fixed by D-term potential in the presence of Fayet-Iliopoulos term as discussed in the case 2:⁹

$$D \propto \xi - |\Phi|^2 \sim 0. \quad (3.16)$$

This implies that Φ multiplet is eaten by the $\text{U}(1)$ gauge multiplet on which we are focusing. Here, we have used the canonical wave function for simplicity and ξ is comparable to the cutoff scale. Thus $\langle \Phi \rangle$ plays a role of heavy mass against vector-like pairs, and they condense as $\langle \Psi^c \Psi \rangle = A^{\frac{1}{N_c}} e^{-\frac{2\pi}{N_c} T} (\vartheta_3(v, q))^{\frac{N_f}{N_c} - 1} \langle \Phi \rangle^{\frac{N_f}{N_c} - 1}$. We integrate out them and obtains low

⁸Dirac operator is given by $D_{\alpha\beta} = \partial + iA_\alpha - iA_\beta$, where A_α denotes the background of gauge potential on the α -th D-brane and so on.

⁹ D will develop the VEV as $D \sim m_{3/2}^2$.

energy effective potential

$$W_{\text{eff}} = N_c A^{\frac{1}{N_c}} e^{-\frac{2\pi}{N_c} T} (\vartheta_3(v, q))^{\frac{N_f}{N_c}} \langle \Phi \rangle^{\frac{N_f}{N_c}} \quad (3.17)$$

$$\equiv \Lambda_{\text{SU}(N_c)}^3 A^{\frac{1}{N_c}} (\vartheta_3(v, q))^{\frac{N_f}{N_c}}. \quad (3.18)$$

Here, $\Lambda_{\text{SU}(N_c)}$ is defined as the dynamically generated scale in the pure $\text{SU}(N_c)$ Yang-Mills theory in the low energy. Thus, low energy potential can be rewritten as $V \sim m_{3/2} \Lambda_{\text{SU}(N_c)}^3 (\vartheta_3(v, q))^{\frac{N_f}{N_c}}$ through the gaugino mass and the trilinear scalar term between Φ and $\Psi^c + \Psi$, if A does not depend on v . We expect, however, that two stacks of flavor branes can give an effect on the worldvolume of strong coupling brane, hence we will find $A \sim [\vartheta_1(v_\alpha, \tilde{q})]^{n_\alpha} [\vartheta_1(v_\beta, \tilde{q})]^{n_\beta}$, where $\tilde{q} = e^{i\pi\tau}$, n_α and n_β denote the number of α -th and β -th D-branes for flavors: two ϑ_1 terms come from backreaction of α -th and β -th D-branes to the γ -th brane. Thus the scalar potential is

$$V \sim m_{3/2} \Lambda_{\text{SU}(N_c)}^3 \left[\vartheta_1 \left(\frac{v}{2N_f}, \tilde{q} \right) \right]^{\frac{2}{N_c}} [\vartheta_3(v, q)]^{\frac{N_f}{N_c}} + \mathcal{O}(\Lambda_{\text{SU}(N_c)}^6). \quad (3.19)$$

Here we have assumed that $v_\alpha + v_\beta$ is stabilized by closed string fluxes around the origin, $v/N_f = v_\alpha - v_\beta$, $\vartheta_1(-v, q) = -\vartheta_1(v, q)$, and we took $n_\alpha = n_\beta = 1$. This potential is similar to the case 1, but now it is accompanied with the ϑ_3 -dependent term. Note that the decay constant is enhanced by $2N_f$ in the ϑ_1 term compared to that in the ϑ_3 term.

3.3 Potential with a gauge coupling depending on the Dedekind eta function

Now let us consider inflation models involving the Dedekind eta function, in which a complex structure modulus τ is an inflaton candidate [41] and a running spectral index is generated naturally. See also refs. [90, 91] and [92] in Heterotic cases. We choose closed string fluxes such that $\partial_\tau \int G \wedge \Omega = 0$ in eq. (3.5) with assumption that all open string moduli are stabilized. It is known that the a holomorphic gauge coupling includes the Dedekind eta function [34, 56]:

$$f_{\text{one-loop}} \supset -\frac{b}{2\pi} \log(\eta(\tau)). \quad (3.20)$$

Here this is obtained after integrating out open string modes stretched among D-branes which are propagating on a part of the bulk extra dimension and hence b denotes 1-loop coefficient of beta function for the gauge coupling in $\mathcal{N} = 2$ SUSY sector. We expect that, as in the case 1, the gauge coupling can depend on $\vartheta_1(\langle v \rangle, q)$, where $q = e^{i\pi\tau}$, when there is another stack of n D-branes. Including these contributions to the gauge coupling with $f_{\text{tree}} = T$, the gaugino condensation in the pure $\text{SU}(N)$ theory is given as

$$\begin{aligned} W_{\text{non-pert}} &\propto \exp \left[-\frac{2\pi}{N} \left(T - \frac{b}{2\pi} \log(\eta(\tau)) \right) \right] e^{\frac{n}{N} \log(\vartheta_1(\langle v \rangle, q))} \\ &\equiv \Lambda_{\text{SU}(N)}^3 [\eta(\tau)]^{\frac{b}{N}} (\vartheta_1(\langle v \rangle, q))^{\frac{n}{N}}. \end{aligned} \quad (3.21)$$

Here, $\Lambda_{\text{SU}(N)}$ is the dynamically generated scale in the $\text{SU}(N)$. Though the gaugino mass term depending on the gravitino mass, the scalar potential includes

$$V \sim m_{3/2} \Lambda_{\text{SU}(N)}^3 [\eta(\tau)]^{\frac{b}{N}} (\vartheta_1(\langle v \rangle, q))^{\frac{n}{N}} + \text{c.c.} + \mathcal{O}(\Lambda_{\text{SU}(N)}^6). \quad (3.22)$$

In refs. [90, 91] on Heterotic orbifolds, $b/N = -2$ can be obtained with a requirement of the modular invariance along τ direction. If τ is a kind of overall modulus, we find $b/N = -6$.

3.4 Super-Planckian decay constant

The decay constant for the inflaton is expected to be of order the string/Planck scale. It is, however, possible to realize a decay constant larger than the Planck scale by the alignment mechanism in the presence of multiple light moduli [93]. For instance, when there are two Yukawa couplings and two light brane moduli v_1 and v_2 , i.e., $\partial_{v_{1,2}} \int G \wedge \Omega = 0$ and $W_{\text{hidden Yukawa}} = \vartheta_3(I_1 v_1 + I_2 v_2, q) \Psi^3 + \vartheta_3(J_1 v_1 + J_2 v_2, q) \Psi'^3$, an enhanced decay constant can be obtained with $|J_1 - J_2|/I < 1$, $I_1 = I_2 \equiv I$ and $\langle \Psi' \rangle < \langle \Psi \rangle$. The inflaton becomes $\text{Re}(v_1 - v_2)$ and the decay constant is enhanced by a factor of $I/|J_1 - J_2|$. Indeed, we have seen the enhancement of the decay constant by $2N_f$ in the scalar potential in the above case 3.

4 Discussion and conclusions

The Universe must be reheated by the inflaton decay some time after inflation. The inflaton appears in the gauge and Yukawa couplings, and so, it may decay into the SM gauge bosons (or gauginos if kinematically accessible) through the gauge couplings or into the SM matter and the Higgs or right-handed neutrinos and $B - L$ Higgs through Yukawa couplings. For instance, the following operators will be relevant to the reheating process:

$$\frac{\phi}{F_v} (F_{\mu\nu})^2, \quad \frac{\phi}{F_v} y_t \bar{t} t H, \quad \frac{\phi}{F_v} y_N \varphi_{B-L} N N. \quad (4.1)$$

Here, we have taken $\langle \phi \rangle \neq 0$ because the modulus develops a non-zero VEV in general, and the decay constant F_v is comparable to the string scale, which is not necessarily equal to the decay constant in the inflaton potential. $F_{\mu\nu}$, t , H , φ_{B-L} and N denote the SM gauge bosons, top quark, $B - L$ Higgs and right-handed neutrino respectively. The thermal leptogenesis is viable if the reheating temperature is enough high as 10^9 GeV [94]. The last operator will be useful to implement non-thermal leptogenesis, even if the reheating temperature becomes lower [95, 96].

In this paper we have studied an extension of the natural inflation, where the inflaton potential has a periodic property. We have proposed elliptic inflation where the inflaton potential is based on the elliptic functions and related functions. Focusing on the elliptic and theta inflation, we have shown that the predicted n_s and r interpolate from natural inflation to the exponential inflation including R^2 -inflation, where n_s asymptotes to $n_s \sim 1 - 2/N \simeq 0.967$ for $N = 60$. We have also studied the inflation model with the Dedekind eta function and shown that a sizable running of the spectral index can be generated owing to the small modulations on the inflaton potential. For a certain value of the

parameter, it is possible to realize $dn_s/d\ln k \sim -0.01$ which is more or less constant over the CMB scales.

Periodicity often appears in the field theories and the string theories through compactifications. Interestingly, such periodic backgrounds restrict the allowed form of the potential, and as a result, the inflaton potential takes a specific form rather than simple cosine function(s). For instance, toroidal background forces infinite series of cosine functions to add up to give the Jacobi theta functions (cf. eqs. (2.15)–(2.18)). Indeed, the Jacobi theta functions are known to appear in gauge and Yukawa couplings in toroidal compactifications, where the variables are open string and complex structure moduli. We have explored various ways to generate the scalar potential which depends on the Jacobi theta functions. The inflaton is identified with the real component of the lightest open string modulus, and we have discussed the stabilization of the other moduli fields based on a supergravity model in type IIB orientifold with flux compactification.

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